Random Texts Exhibit Zipf's-Law-Like Word Frequency Distribution

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Abstract. It is shown that the distribution of word frequencies for randomly generated texts is very similar to Zipf's law observed in natural languages such as the English. The facts that the frequency of occurrence of a word is almost an inverse power law function of its rank and the exponent of this inverse power law is very close to 1 are largely due to the transformation from the word's length to its rank, which stretches an exponential function to a power law function.

Zipf observed long time ago [1] that the distribution of word frequencies in English, if the words are aligned according to their ranks, is an inverse power law with the exponent very close to 1. In other words, if the most frequently occurring word appears in the text with the frequency $P(1)$, the next most frequently occurring word has the frequency $P(2)$, and the rank-r word has the frequency $P(r)$, the frequency distribution is

$$P(r) = \frac{C}{r^\alpha}, \tag{0.1}$$

with $C \approx 0.1$ and $\alpha \approx 1$. This distribution, also called Zipf's law, has been checked for accuracy for the standard corpus of the present-day English with very good results [2].

The fall-off of the distribution as the rank is increased is obvious, because the more frequently occurring words are guaranteed to have larger frequencies than those less frequently occurring. Nevertheless, it seems to be a puzzle as why the decay is a power law instead of an exponential function or other faster decaying functions, and why the exponent is very close to 1 instead of 2 or even larger values. There are attempts to incorporate Zipf's law into the grander framework of "fractals" [3], but in doing so, little insight has been gained in understanding this particular "law."

Probably few people pay attention to a comment by Miller in his preface to Zipf's book [4], that randomly generated texts, which are perhaps the least interesting sequences and unrelated to any other scaling behaviors, also exhibit Zipf's law. What he said was that Zipf's law is not exclusive for English or any other natural languages. Miller did not give

a proof of his statement, and it is the purpose of this short paper to provide a very simple proof that random texts do indeed exhibit Zipf's-law-like word frequency distribution.

By "random texts", we mean the symbolic sequences generated by the following procedure: each symbol out of total $(M + 1)$ symbols is selected randomly and deposited at position i, and another symbol is randomly selected and deposited at position $i + 1$, and so on. There is no correlation between the selection of symbol at position i and that at position $i + 1$. Among the $(M + 1)$ symbols, one of them is called the "blank space". Any "non-blank" symbol string between two blank spaces is called a "word," whereas a string of blank spaces is not. Taking the English alphabets for example, M=26, and the words in random texts can be *a, b, c, ..., aa, ab, ac, ..., ba, bb, ..., aaa, aab ...,* etc. If the following sequence, for example, is generated,

$$a\_mdf\_pwell\_werlppa\_re\_\_kkel\_,$$

it then contains the words *a* (suppose that the beginning of the sequence also plays the role of a blank space), *mdf, pwell, werlppa, re,* and *kkel.*

The probability that one would see the string $\_a\_$ in a random text is $(1/27)^3$, which is equal to the product of the probability for the first symbol to be a blank space ($=1/27$), for the second symbol to be a ($=1/27$), and for the third symbol to be a blank space ($=1/27$). Similarly, the probability for finding the string $\_bsl\_$ is $(1/27)^5$. Since the first probability is also the frequency of occurrence for any word with length 1 (except a normalization factor), and the second probability is the frequency of occurrence for any word with length 3, we have the general formula for the frequency of occurrence for any word with length L:

$$P_i(L) = c\frac{1}{(M + 1)^{L+2}}, \quad i=1,2, \dots M^L. \tag{0.2}$$

Note that there are M^L words having length L.

The constant c can be determined from the normalization condition for the frequencies of occurrence of all words:

$$\sum_{L=1}^{\infty} M^L \frac{c}{(M + 1)^{L+2}} = c\frac{M}{(M + 1)^2} = 1, \tag{0.3}$$

so

$$c = \frac{(M + 1)^2}{M}. \tag{0.4}$$

Inserting the value of c back to the Eq. (0.2), the frequency of occurrence for any particular word with length L is

$$P_i(L) = \frac{1}{M(M + 1)^L} \tag{0.5}$$

and the frequency of occurrence for all words with length L is

$$P(L) = M^L P_i(L) = \frac{M^{L-1}}{(M + 1)^L}. \tag{0.6}$$

Both are exponential functions of L.

In a random text, all words with the length L rank higher than words with the length $L + 1$, because they have larger value of frequency of occurrence by Eq. (0.5). If we represent the rank of any word with length L by $r(L)$, we have:

$$\sum_{l=1}^{L-1} M^l < r(L) \le \sum_{l=1}^{L} M^l \tag{0.7}$$

or

$$\frac{M}{M-1}(M^{L-1} - 1) < r(L) \le \frac{M}{M-1}(M^L - 1). \tag{0.8}$$

For example, $1 < r(1) \le M$, $M < r(2) \le M + M^2$, and so on. Eq. (0.8) represents the exponential transformation from word's length to word's rank. One implication of the transformation to be exponential is that the longer the L, the more "stretching" of the rank variable, since there are more number of words with longer lengths.

The Eq. (0.8) can be converted to

$$L - 1 < \log_M \left(\frac{M-1}{M} r(L) + 1 \right) \le L. \tag{0.9}$$

Raising all the terms to the power of $1/(M + 1)$:

$$\frac{1}{(M+1)^{L-1}} > \left(\frac{1}{M+1} \right)^{\log_M \left(\frac{M-1}{M} r(L) + 1 \right)} \ge \frac{1}{(M+1)^L}, \tag{0.10}$$

multiplying all terms by $1/M$,

$$\frac{1}{M(M+1)^{L-1}} > \frac{1}{M} \left(\frac{1}{\frac{M-1}{M} r(L) + 1} \right)^{\frac{\log(M+1)}{\log(M)}} \ge \frac{1}{M(M+1)^L}, \tag{0.11}$$

which can be written as:

$$P_i(L) < \frac{C}{(r(L) + B)^\alpha} \le P_i(L - 1) \tag{0.12}$$

with

$$\alpha = \frac{\log(M+1)}{\log(M)}, \quad B = \frac{M}{M-1}, \quad \text{and} \quad C = \frac{1}{M} \frac{M^\alpha}{(M-1)^\alpha} = \frac{M^{\alpha-1}}{(M-1)^\alpha}. \tag{0.13}$$

The functional form

$$P(r) = \frac{C}{(r + B)^\alpha} \tag{0.14}$$

is also called the generalized Zipf's law by Mandelbrot [3, 5]. Let us check how close the generalized Zipf's law for random texts can be to the Zipf's law in English: since the number of alphabets is $M = 26$, we have $\alpha = 1.01158$ and $C = 0.04$. The exponent

α is extremely close to what is observed in English, an amazing fact considering how little we have assumed. Even with the minimum number of symbols, $M = 2$ (if $M = 1$, the transformation from the word length to the word rank is linear, and no power-law distribution is expected), $\alpha = 1.58496$ is still not that far from 1.

The frequency of occurrence of words by their rank represented by Eq. (0.12) does not have the problem of divergence of the total probability typical for a power-law distribution, because the exponent $\alpha = 1.01158$ is strictly larger than 1 — which takes care of the integration at the tail end; and there is a cutoff of the smallest word rank, i.e., $r = 1$ — which takes care of the integration at the zero value of the rank.

Due to the assumption that each symbol appears in the sequence with exactly the same probability, all words with the same length have the same frequency of occurrence. In other words, $P(r)$ is a stepwise function having plateaus on $P_i(L)$'s. Figure 1 shows a numerical result of the word frequency distributions for random texts with 2, 4, and 6 symbols, respectively. In the numerical simulation, a sequence of length N (which is 80000, 200000, and 600000 for $M=2$, 4 and 6) is generated with the M symbols and the blank space all having the equal probability. I also introduce a cutoff of the maximum possible word length L_{max} (6, 4 and 3 for $M=2$, 4 and 6). The frequency of a word is derived by dividing the number of occurrence of that word with the total number of word countings (which is 16306, 18964, and 36320 for $M=2$, 4 and 6). Since I do not count words whose lengths are longer than the cutoff length, the normalization condition Eq. (0.3) now becomes:

$$\sum_{L=1}^{L_{max}} M^L \frac{c}{(M+1)^{L+2}} = \frac{c}{(M+1)^2} M \left(1 - \left(\frac{M}{M+1}\right)^{L_{max}}\right) = 1, \qquad (0.15)$$

which leads to a larger value of c

$$c = \frac{(M+1)^2}{M} \frac{1}{1 - (\frac{M}{M+1})^{L_{max}}} \qquad (0.16)$$

but the α estimated by Eq. (0.13) should be the same. To make a comparison with Zipf's law ($\sim 1/r$) as well as the power law with the exponent 2 ($\sim 1/r^2$), these two functions are also plotted in Fig. 1. The numerical simulation confirms that the random texts exhibit a word frequency distribution very much the same with Zipf's law.

It is clear now that the existence of the Zipf's-law-like word frequency distribution in random texts is purely due to the choice of the rank as the independent variable. By choosing the word rank rather than the word length, the exponential distribution which is typical for random texts becomes a power law function. This strongly suggests that the power law as expressed by Zipf's law in natural languages is also purely due to the choice of the rank as the independent variable. Actually, besides the cardinal number and the ordinal number, one can also use the third representation of the same frequency distribution:

the distribution on certain position of the digit, as related to the notorious "first digit problem"[6]. The transformation among the three representations is summarized by Gell-Mann [7].

Equations (0.13) also explains why the exponent in Zipf's law is close to 1, simply because $\log(M+1) \approx \log(M)$ when M is large. As we have seen, even for the worse case of having two symbols ($M = 2$), the estimated $\alpha \approx 1.58$ is still smaller than 2. Only for $M=1$ (the sequence is a binary sequence with one symbol and one blank space), no mechanism exists for stretching the frequency distribution from exponential to power law, and we fail to recover Zipf's law. If Zipf's law is observed for binary sequences, it indicates a "true" power law scaling, and one should expect other non-trivial scaling behaviors, such as the 1/f noise and the long-range correlations [8].

The stepwise structure of the frequency of occurrence distribution in Fig.1 can be removed by introducing bias among different symbols, i.e., different symbols have different probabilities to appear in the sequence. For example, symbol a can be more likely to appear in the sequence than symbol b; and consequently, word _a_ has a larger value of the frequency of occurrence than word _b_. The plateaus are then easily destroyed. In particular, a word with longer length can have a larger frequency than the words with shorter lengths; for example, word _aa_ ranks higher than word _b_ if the square of the probability for symbol a to appear in the sequence is larger than the probability for symbol b. Figure 2 shows the numerical results for two biased random sequences with two and four symbols respectively. For the two-symbol sequence, I choose the probability for having blank space to be 0.33, the probability for the first symbol is 0.47 and that for the second symbol is 0.2 (these numbers are arbitrarily chosen). For the four-symbol sequence, the probability for the blank space is 0.2, those for the remaining symbols are 0.5, 0.13, 0.1, and 0.07 (again, those are arbitrary numbers). A much smoother power law distributions show up in Fig.2.

In conclusion, Zipf's law is not a deep law in natural language as one might first have thought. It is very much related the particular representation one chooses, i.e., rank as the independent variable. A symbolic sequence which exhibits Zipf's law does not have to exhibit other scaling phenomena such as the 1/f noise or long-range correlation. In fact, the long-range correlation and 1/f spectrum are absent in natural languages, as having been observed by the author that the mutual information function between two letters decays faster than power laws of small exponents [9]. Mandelbrot [3] seems to derive the same result that random texts exhibit the generalized Zipf's law by using lexicographic trees, and noticed that Zipf's law is "linguistically very shallow." But he still tries to link Zipf's law with other scaling phenomena. This paper provides a much intuitive derivation and emphasizes that Zipf's law does not share the common ground with other scaling behaviors.

Acknowledgements

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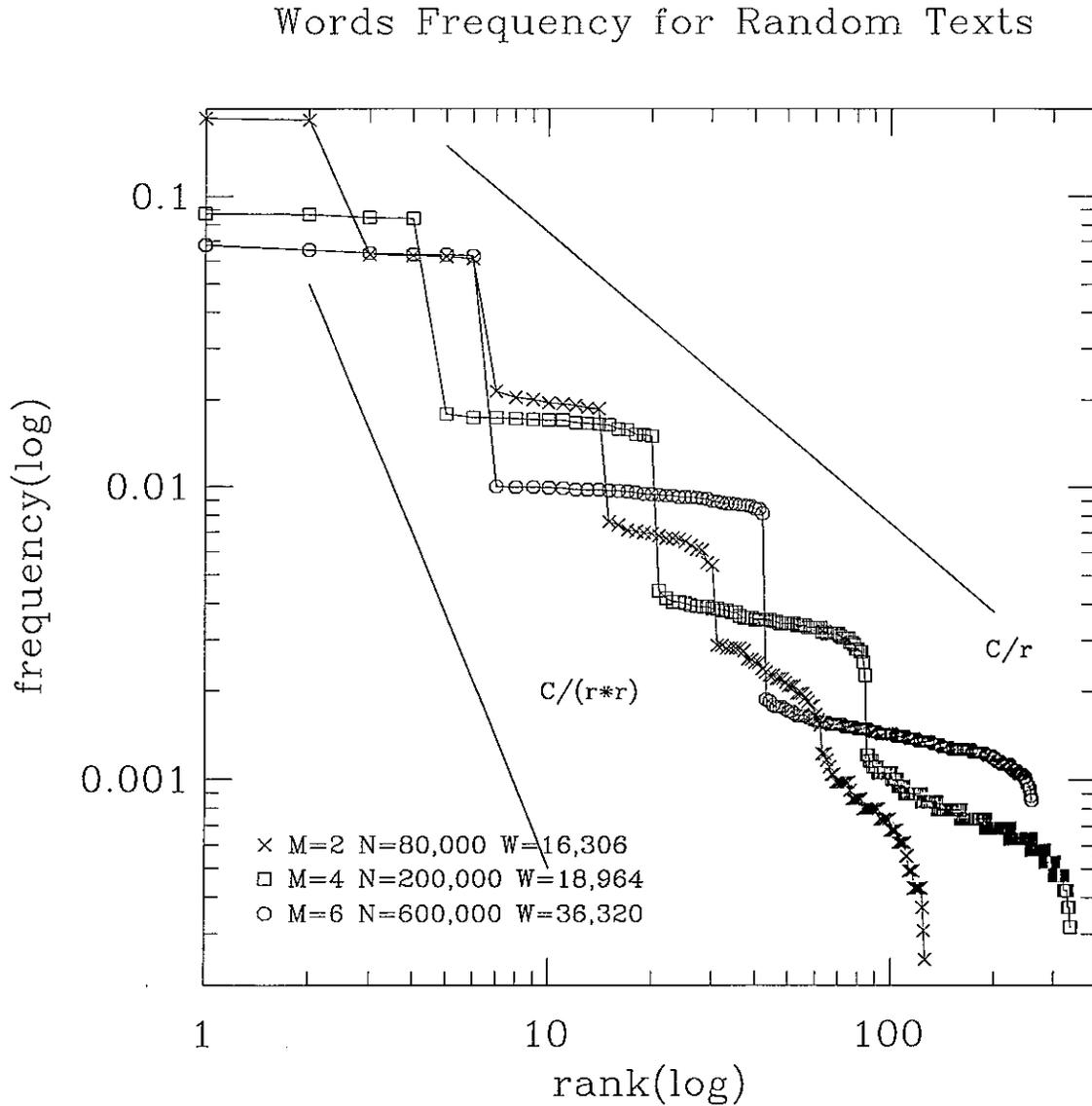
Words Frequency for Random Texts

Figure 1: Word frequency as the function of the word's rank for randomly generated sequences with the number of symbols $M=2$, 4, and 6. There is a cutoff for the longest word length to be counted (the cutoffs *Lmax* are 6, 4 and 3 respectively for $M=2$, 4, and 6). All symbols including the blank space have the same probability to appear in the sequence. The frequency of occurrence of a word is the number of countings of that word divided by the number of countings of all words (they are 16306, 18964, and 36320 respectively for $M=2$, 4, and 6). Also shown are Zipf's scaling law (power law function with the exponent 1) and the power law function with the exponent 2.
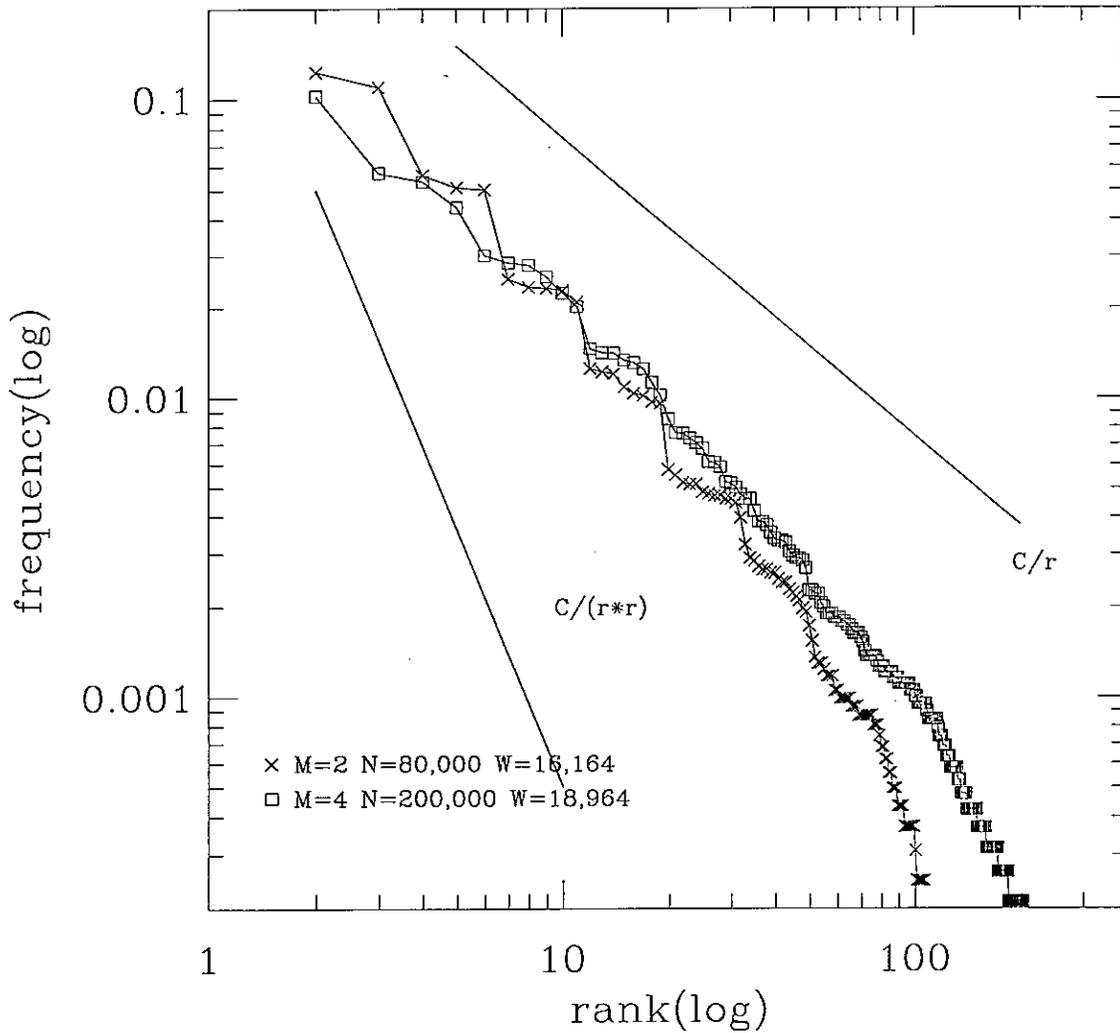
Words Frequency for Bias Random Texts

Figure 2: Word frequency as the function of the word's rank for biased random sequences with the number of symbols $M=2$ and 4. There is a cutoff for the longest word length to be counted ($Lmax = 6$ and 4 respectively for $M=2$ and 4). Different symbols as well as the blank space have different probability to appear in the sequence (see the text for their values). The frequency of occurrence of a word is the number of countings of that word divided by the number of countings of all words (they are 16164 and 18964 respectively for $M=2$ and 4). Also shown are Zipf's scaling law (power law function with the exponent 1) and the power law function with the exponent 2.